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Permutation Games: Another Class of Totally Balanced Games

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Summary. A class of cooperative games in characteristic function form arising from certain sequencing problems and assignment problems, is introduced. It is shown that games of this class are totally balanced. In the proof of this fact we use the Birkhoff-von Neumann theorem on doubly stochastic matrices and the Bondareva-Shapley theorem on balanced games. It turns out that this class of permutation games coincides with the class of totally balanced games if the number of players is smaller than four. For larger games the class of permutation games is a nonconvex subset of the convex cone of totally balanced games.

Zusammenfassung. Wir führen eine Klasse von kooperativen Spielen in charakteristischer Funktionsform ein, die bei gewissen Folgeproblemen und Zuordnungsproblemen auftreten. Wir zeigen, daß diese Spiele vollständig balanciert sind. Zum Beweis verwenden wir den Satz von Birkhoff-von Neumann über doppelt stochastische Matrizen und den Satz von Bondareva-Shaplex über balancierte Spiele. Es zeigt sich, daß diese Klasse von Permutationsspielen mit der Klasse von vollständig balancierten Spiele übereinstimmt, falls die Zahl der Spieler kleiner als vier ist. Für größere Spiele ist die Klasse der Permutationsspiele eine nichtkonvexe Teilmenge des konvexen Kegels der vollständig balancierten Spiele.

(iii) the customers are lined up before the counter in the order $1, 2, \dots, n$ i.e., a customer i , $1 \leq i \leq n$, gets his turn for service during the period $[(i-1)t, it]$ if there is no cooperation.

(iv) Subsets of customers are allowed to rearrange their positions before the counter and also sidepayments between them are allowed.

(v) If customer i , $1 \leq i \leq n$, gets his turn for service during the period $[(j-1)t, jt]$, $1 \leq j \leq n$, then he has to incur cost k_{ij} .

In general by cooperating the customers can reduce their total costs by rearranging their positions before the counter. Therefore this situation can be reduced to a cooperative game with player set $N := \{1, 2, \dots, n\}$ and with the characteristic function $c : 2^N \rightarrow \mathbb{R}$ with $c(\emptyset) = 0$ and where for each non-empty coalition $S \in 2^N$, $c(S)$ denotes the minimal total costs of that coalition in co-operating. Clearly for each $S \in 2^N - \{\emptyset\}$ we have

$$c(S) = \min_{\pi} \sum_{i \in S} k_{i\pi(i)} \quad (1.1)$$

with the minimum taken over all S -permutations $\pi : S \rightarrow S$, where π corresponds to that arrangement of positions before the counter, where customer i gets his turn for service during the period $[(\pi(i)-1)t, \pi(i)t]$.

Let us now consider another situation.

(i) There are n persons $1, 2, \dots, n$ and person i , $1 \leq i \leq n$, possesses a machine M_i and has a job J_i to be processed.

(ii) Any machine M_j can process any job J_i , but no machine is allowed to process more than one job.

(iii) Coalition formation and sidepayments are allowed.

(iv) If a person does not cooperate, his job has to be processed on his own machine.

1. A Sequencing Problem and an Assignment Problem

Let us consider the following situation, where

(i) there are n customers $1, 2, \dots, n$ waiting for service at a counter.

(ii) for each customer the service time is equal, say, t .

(v) The cost of processing job J_i on machine M_j equals k_{ij} , where $1 \leq i, j \leq n$.

This situation can also be reduced to the n -person game with characteristic function $c : 2^N \rightarrow \mathbb{R}$, described by (1.1), but now an S -permutation $\pi : S \rightarrow S$ corresponds to the assignment of the jobs of coalition S to the machines of S , in such a way that machine $M_{\pi(i)}$ processes job J_i . Games of the above type will be called permutation games. Formally, we have

Definition 1.1. An n -person game $\langle N, c \rangle$ in characteristic function form is a *permutation game*, if there exists a (cost) matrix $K = [k_{ij}]_{i=1, j=1}^n$ such that $c(S)$ is given by (1.1) for each $S \in 2^N - \{\emptyset\}$.

Let us give the characteristic functions of two permutation games which play also a role in the proof of theorem 3.3.

Example 1.2. Let $\langle N, c^1 \rangle$ and $\langle N, c^2 \rangle$ be the 4-person permutation games with cost matrices

$$K^1 := \begin{bmatrix} 0 & 0 & 0 & -1 \\ -2 & 0 & 0 & -1 \\ -2 & -2 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad K^2 := \begin{bmatrix} 0 & 2 & 0 & 4 \\ -4 & 0 & 0 & 2 \\ -2 & -2 & 0 & 2 \\ -6 & -4 & -4 & 0 \end{bmatrix}$$

respectively. Then we have

$$\begin{aligned} c^1(S) &= c^2(S) = 0 & \text{if } |S| \leq 1, \\ c^1(S) &= c^2(S) = -2 & \text{if } |S| = 2, \\ c^1(S) &= c^2(S) = -4 & \text{if } S \neq \{1, 2, 3\} \text{ and } |S| = 3, \\ c^1(\{1, 2, 3\}) &= -4, \quad c^2(\{1, 2, 3\}) = -6, \\ c^1(N) &= c^2(N) = -6. \end{aligned}$$

The main question which we answer in this paper is: has a permutation game $\langle N, c \rangle$ a non-empty core? Here the core of $\langle N, c \rangle$ is defined by

$$\begin{aligned} \text{Core}(c) &:= \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = c(N), \\ &\quad \sum_{i \in S} x_i \leq c(S) \text{ for all } S \in 2^N\}. \end{aligned}$$

A core element $x = (x_1, x_2, \dots, x_n)$ can be seen as an allocation of the total costs $c(N)$ of the grand coalition N to the players, where player i has to contribute x_i . Furthermore, since $\sum_{i \in S} x_i \leq c(S)$, no subset S of players can decrease its total costs by forming a subcoalition. So core elements are in this sense stable cost allocations and it is therefore important to know whether the core of a permutation game is non-empty. Note that $\text{Core}(c^2) = \{(-2, -2, -2, 0)\}$ and $(-1\frac{1}{2}, -1\frac{1}{2}, -1\frac{1}{2}, -1\frac{1}{2}) \in \text{Core}(c^1)$,

if c^1 and c^2 are as in Example 1.2. In the next section we prove that all permutation games possess a nonempty core.

In Sect. 3 we answer the question whether each totally balanced game can be seen as a permutation game. In the last section some remarks are made on papers which are in the same line of research as this paper.

2. The Core of Permutation Games

In this section we answer the main question, posed in Sect. 1, in the affirmative. We start with some notation. Let $T_{n \times n}$ be the set of $n \times n$ -matrices $A = [a_{ij}]_{i=1, j=1}^n$ with $a_{ij} \in \{0, 1\}$ for all i, j . Let for each $S \in 2^N - \{\emptyset\}$, e^S be the vector in \mathbb{R}^n with i -th coordinate $e_i^S = 0$ if $i \notin S$ and $e_i^S = 1$ if $i \in S$. Let us denote by $P(S)$ the set of S -permutation matrices. Hence

$$P(S) := \{R \in T_{n \times n} : e^N R = e^S, R e^N = e^S\}.$$

We note that each S -permutation $\pi : S \rightarrow S$ corresponds in a natural manner to the S -permutation matrix $[r_{i,j}^\pi]_{i=1, j=1}^n$, where $r_{i,j}^\pi = 1$ iff $j = \pi(i)$. Let us denote the inner product $\sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij}$ of two real $n \times n$ -matrices

$$A = [a_{ij}]_{i=1, j=1}^n \quad \text{and} \quad B = [b_{ij}]_{i=1, j=1}^n \quad \text{by} \quad A * B.$$

Then we can rewrite (1.1) as follows:

$$c(S) = \min_{R \in P(S)} K * R \quad (2.1)$$

where K is the cost matrix $[k_{ij}]_{i=1, j=1}^n$.

In the following, elements of $P(N)$ are called permutation matrices and real $n \times n$ -matrices D with

$$D \geq 0, \quad e^N D = e^N, \quad D e^N = e^N$$

are called doubly stochastic matrices.

We use in the proof of Theorem 2.3 the following result due to Birkhoff [2] and von Neumann [8].

Theorem 2.1. Each doubly stochastic $n \times n$ -matrix is a convex combination of permutation matrices.

Also use will be made of the following result of Bondareva [3] and Shapley [10], which is a consequence of the duality theorem for LP-problems.

Theorem 2.2. Let $\langle N, c \rangle$ be a cooperative n -person game. Then $\text{Core}(c) \neq \emptyset$ if and only if for each balanced family $B \subset 2^N - \{\emptyset\}$ with positive weights $(\lambda_S)_{S \in B}$ i.e.

$$e^N = \sum_{S \in B} \lambda_S e^S \text{ and } \lambda_S > 0 \text{ for all } S \in B \quad (2.2)$$

we have

$$\sum_{S \in B} \lambda_S c(S) \geq c(N) \quad (2.3)$$

Now we state and prove our main result.

Theorem 2.3. *Each permutation game has a non-empty core.*

Proof. Take a permutation game $\langle N, c \rangle$ with cost matrix K and let $B \subset 2^N$ be a balanced family with weights $(\lambda_S)_{S \in B}$. In view of Theorem 2.2 we have only to show that (2.3) holds, knowing (2.2). In view of (2.1), for each $S \in 2^N - \{\emptyset\}$ we can take an $M_S \in P(S)$ such that

$$K * M_S = \min_{R \in P(S)} K * R = c(S) \quad (2.4)$$

Define the $n \times n$ -matrix D by $D := \sum_{S \in B} \lambda_S M_S$ and note that

$$D \geq 0, \quad e^N D = \sum_{S \in B} \lambda_S e^N M_S = \sum_{S \in B} \lambda_S e^S = e^N$$

because $M_S \in P(S)$ and because (2.2) holds. Similarly, $D e^N = e^N$. Hence, D is a doubly stochastic matrix. By Theorem 2.1 there exist $k \in N$, non-negative real numbers $\beta_1, \beta_2, \dots, \beta_k$ with $\sum_{i=1}^k \beta_i = 1$ and permutation matrices R_1, R_2, \dots, R_k such that $D = \sum_{i=1}^k \beta_i R_i$. But then, by (2.4) with N in the role of S :

$$K * D = \sum_{i=1}^k \beta_i K * R_i \geq \sum_{i=1}^k \beta_i c(N) = c(N) \quad (2.5)$$

since also by (2.4):

$$\sum_{S \in B} \lambda_S c(S) = \sum_{S \in B} \lambda_S K * M_S = K * D, \quad (2.6)$$

we can conclude from (2.5) and (2.6) that (2.3) holds, which finishes the proof. \square

3. Totally Balanced Games

A game $\langle N, c \rangle$ satisfying (2.3) for each balanced family B is called a *balanced game* and a game for which also each subgame is balanced, is called a *totally balanced game*. Since each subgame of a permutation game is again a

permutation game, we can conclude from Theorem 2.2 and 2.3:

Theorem 3.1. *Each permutation game is totally balanced.*

An interesting question is: Can each totally balanced game be seen as a permutation game?

In Theorem 3.2 we prove that two and three-person totally balanced games indeed correspond to permutation games. For more than three players the family of permutation games turns out to be a non-convex (and therefore proper) subset of the convex cone of totally balanced games as we can conclude from Theorem 3.3.

Theorem 3.2. *Let $\langle N, c \rangle$ be a totally balanced game and $n \in \{2, 3\}$. Then there exists an $n \times n$ -matrix K such that c satisfies (2.1).*

Proof. Without loss of generality we suppose that $\langle N, c \rangle$ is zero-normalized i.e. $c(i) = 0$ for $i \in N$.

If $n = 2$, take

$$K := \begin{bmatrix} 0 & 0 \\ c(N) & 0 \end{bmatrix}$$

and if $n = 3$, take

$$K := \begin{bmatrix} 0 & 0 & \alpha \\ c(1, 2) & 0 & 0 \\ \beta & c(2, 3) & 0 \end{bmatrix}$$

where $\alpha := c(N) - c(1, 2) - c(2, 3)$ and $\beta = c(1, 3) - \alpha$. It is straightforward to show that (2.1) holds. \square

Theorem 3.3. *Let PG^n be the family of n -person permutation games. Then for $n \geq 4$, PG^n is not a convex set.*

Proof. (i) First we prove the theorem for $n = 4$. Take the permutation games $\langle N, c^1 \rangle$ and $\langle N, c^2 \rangle$ of Example 1.2. Let $c := \frac{1}{2}(c^1 + c^2)$. Then $\langle N, c \rangle$ is totally balanced, because by theorem 3.1, c^1 and c^2 are totally balanced and because the family of characteristic functions of totally balanced n -person games forms a convex cone. We show that $\langle N, c \rangle$ is not a permutation game. Suppose for a moment that there is a cost matrix $K = [k_{ij}]_{i,j=1}^4$ such that the corresponding permutation game is $\langle N, c \rangle$. Then

$$k_{ii} = c(\{i\}) = 0 \quad \text{for all } i \in \{1, 2, 3, 4\}, \quad (3.1)$$

$$k_{ij} + k_{ji} = c(\{i, j\}) = -2 \quad \text{for all } i, j \in \{1, 2, 3, 4\}. \quad (3.2)$$

In view of (3.1) and (3.2) we obtain: $-5 = c(\{1, 2, 3\})$

$$= \min_{\pi \in P\{1,2,3\}} \sum_{i=1}^3 k_{i\pi(i)} \\ = \min \{0, k_{12} + k_{23} + k_{31}, -6 - (k_{12} + k_{23} + k_{31}), -2, -2, -2\}.$$

Hence, $-5 = k_{12} + k_{23} + k_{31}$ or $-5 = -6 - (k_{12} + k_{23} + k_{31})$, so

$$k_{12} + k_{23} + k_{31} \in \{-5, -1\}. \quad (3.3)$$

Similarly, by considering $c(\{1, 2, 3\})$, $c(\{1, 3, 4\})$ and $c(\{2, 3, 4\})$, respectively, we obtain

$$k_{12} + k_{24} + k_{41} \in \{-4, -2\}, \quad (3.4)$$

$$k_{13} + k_{34} + k_{41} \in \{-4, -2\}, \quad (3.5)$$

$$k_{23} + k_{34} + k_{42} \in \{-4, -2\}. \quad (3.6)$$

Adding the left hand sides of (3.3) and (3.5) and subtracting from this sum the left hand sides of (3.4) and (3.6), and using (3.2), we obtain as result 0. Hence,

$$0 \in \{-5, -1\} + \{-4, -2\} - \{-4, -2\} - \{-4, -2\} \quad (3.7)$$

and that is impossible because 0 is even and the set on the right side of (3.7) contains only odd numbers.

Hence, the totally balanced game $\langle N, c \rangle$ is no permutation game. This proves that PG^4 is not convex.

(ii) Now let $n > 4$. Then we construct games $\langle N, c_n^1 \rangle$ and $\langle N, c_n^2 \rangle$ in PG^n with the aid of c^1 and c^2 of example 1.2 by adding dummy players. Formally, let $M := \{1, 2, 3, 4\}$, then take

$$c_n^1(S) := c^1(S \cap M),$$

$$c_n^2(S) := c^2(S \cap M) \quad \text{for all } S \subset N.$$

Now $\langle N, c_n^1 \rangle$ and $\langle N, c_n^2 \rangle$ are permutation games with cost matrices K_n^1 and K_n^2 , where for $k \in \{1, 2\}$:

$$(K_n^k)_{ij} := (K^k)_{ij} \quad \text{if } i, j \in M,$$

$$(K_n^k)_{ii} := 0 \quad \text{for all } i \in N \quad \text{and}$$

$$(K_n^k)_{ij} := \lambda \quad \text{if } i \neq j \text{ and } i \notin M \text{ or } j \notin M,$$

where λ is a very large number and K^k is the cost matrix corresponding to c^k . Similarly, as in (i) one shows that $\frac{1}{2}(c_n^1 + c_n^2) \notin PG^n$. \square

4. Conclusions and Remarks

We have seen that certain sequencing and assignment situations give rise in a natural way to the class of permutation games. We have shown that each permutation game is totally balanced but that the converse is not true if the number of players is at least four.

There is now a substantial literature on economic situations which give rise to totally balanced games. We want to pay some attention to some of these papers which are in the same line of research as this paper.

Shapley and Shubik [13] introduce a class of market games and prove that each market game is totally balanced and, conversely, that each totally balanced game corresponds to a market game (cf. Billera [1]).

Granot and Huberman [5] consider the problem of cost allocation among users of a minimum cost spanning tree network. They prove that the minimum cost spanning tree games, corresponding to such problems, have a non-empty core.

Kalai and Zemel [6] show that flow games are totally balanced and that each totally balanced game corresponds also to a flow game.

Owen [9] proves that linear programming games have a non-empty core. In Kalai and Zemel [7] and in Dubey and Shapley [4] sufficient conditions are given for controlled mathematical programming problems to guarantee that the corresponding games are balanced.

Shapley and Scarf [12] consider a market with n traders, each with an indivisible good to offer in trade. The goods are freely transferable, but a trader never needs more than one item. Contrary to the situations in Sect. 1, they suppose that there is no money or other medium of exchange. Their main result is that the corresponding game *without* sidepayments has a non-empty core.

Finally, we want to pay some attention to assignment games, introduced in Shapley and Shubik [14]. Such a game arises as follows. There is a set $B = \{1, 2, \dots, n\}$ of buyers and a set $B' = \{1', 2', \dots, m'\}$ of sellers of a commodity and a non-negative reward matrix $A = [a_{ij}]_{i=1, j'=1'}^n$. For each $S \subset B \cup B'$, let $u(S) := a_{ij}$, if $S = \{i, j'\}$ and let $u(S) := 0$, otherwise. Let v be the superadditive cover of u . Then $\langle B \cup B', v \rangle$ is the assignment game, corresponding to the reward matrix A . Shapley and Shubik [14] proved that such an assignment game has non-empty reward core and that all core elements can be obtained as solutions of the linear programming problem dual to the optimal assignment problem. A main difference between assignment games and permutation games is that in situations giving rise to permutation games a player is simultaneously a buyer as well as a seller.

Nevertheless it is possible, as Shapley [11] showed to us in a letter, to find a connection and to exploit this

connection to find another proof of Theorem 2.3. We will sketch the way Shapley indicated to go.

(i) Let $\langle N, c \rangle$ be a permutation game with cost matrix $K = [k_{ij}]_{i=1, j=1}^n$ and let $M := \max \{k_{ij} : i, j \in N\}$.

Look at the assignment game $\langle B \cup B', v \rangle$ with $B = N = \{1, 2, \dots, n\}$ and $B' = N' = \{1', 2', \dots, n'\}$ and with reward matrix $A = [M - k_{ij'}]_{i=1, j'=1}^n$.

(ii) One can show that for all $S \subset N$:

$$c(S) = |S| M - v(S \cup S') \quad (4.1)$$

where $S' = \{i' \in N' : i \in S\}$ and $|S|$ is the number of elements in S . Furthermore, (4.1) implies: if $(x_1, \dots, x_n, x'_1, \dots, x'_n)$ is an element of the (reward) core of $\langle N, v \rangle$, then

$$\begin{aligned} z &:= (M - (x_1 + x'_1), M - (x_2 + x'_2), \dots, M - (x_n + x'_n)) \\ &\in \text{core}(c) \end{aligned} \quad (4.2)$$

Since the (reward) core of $\langle N, v \rangle$ is non-empty, from (4.2) we may again conclude that $\text{core}(c) \neq \emptyset$.

Addendum. One of the referees indicated that another proof of our Theorem 2.3 can be given using the Birkhoff-von Neumann theorem and Theorem 1 of Owen [9]. The idea is to embed the set of permutation games into the set of linear production games. In order to accomplish that embedding we need the following observations.

(i) We lose no generality if we restrict ourselves to non-positive cost matrices K .

(ii) From the Birkhoff-von Neumann theorem and $K \leq 0$ it follows (cf. [8], Lemma 1) that

$$c(S) = \min_{R \in P(S)} K * R = \min_{D \in D^+(S)} K * D$$

where $D^+(S)$ is the set of sub-doubly stochastic $n \times n$ -matrices with respect to S .

(iii) The set $D^+(S)$ can be described as follows. Let A_{2i-1} be the zero-one $n \times n$ -matrix whose i -th row is $(1, 1, \dots, 1)$ and with zeros elsewhere and let A_{2i} be the

zero-one $n \times n$ matrix with ones in the i -th column and zeros elsewhere. Furthermore, let $b^i \in \mathbb{R}^{2n}$ be the vector with 1 as $(2i-1)$ -th and as $2i$ -th coordinate and with all other coordinates zero. Then $X \in D^+(S)$ iff $A_k * X \leq (\sum_{j \in S} b^j)_k$ for all $k \in \{1, 2, \dots, 2n\}$ and $X \geq 0$.

Then the permutation game with non-positive cost matrix K is equivalent to the linear production game, for which coalition S is interested in the linear program $\max X * (-K)$ under the restrictions $X \geq 0$ and

$$X * A_k \leq (\sum_{j \in S} b^j)_k \quad \text{for all } k \in \{1, 2, \dots, 2n\}.$$

Such a production game has a non-empty core by Owen's result.

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